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LETTER TO THE EDITOR

Some constant solutions to Zamolodchikov's tetrahedron equations

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Abstract. In this letter we present constant solutions to the tetrahedron equations proposed by Zamolodchikov. In general, from a given solution of the Yang–Baxter equation there are two ways to construct solutions to the tetrahedron equation. There are also other kinds of solutions. We present some two-dimensional solutions that were obtained by directly solving the equations using either an upper triangular or Zamolodchikov's ansatz.

The theory of integrable dynamical systems in 1+1 dimensions (both continuous PDE and discrete lattice systems) is now fairly well understood, and at present an increasing amount of research is focused on generalizations to higher dimensions.

In 1+1 dimensions the key equation for integrability of lattice systems [1] and quantum inverse transformation [2] is the quantum Yang–Baxter equation (YBE) [3]

$$R_{j_1j_2}^{k_1k_2}(u)R_{k_1j_3}^{l_1k_3}(u+v)R_{k_2k_3}^{l_2l_3}(v)=R_{j_2j_3}^{k_2k_3}(v)R_{j_1k_3}^{k_1l_3}(u+v)R_{k_1k_2}^{l_1l_2}(u). \tag{1}$$

(Here and in the following summation over repeated indices is understood.) In dimension N there are N^6 equations for N^4 variables.

The constant, spectral parameter independent version of (1) is

$$R_{j_1j_2}^{k_1k_2}R_{k_1j_3}^{l_1k_3}R_{k_2k_3}^{l_2l_3}=R_{j_2j_3}^{k_2k_3}R_{j_1k_3}^{k_1l_3}R_{k_1k_2}^{l_1l_2}. \tag{2a}$$

This is obtained as the limit $u=v=0$ or $u=v=\pm\infty$, but appears also independently in the study of quantum groups [4] and knot theory [5]. In the shorthand notation one writes out only the labels of subspaces on which the matrices act:

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{2b}$$

In [6] Zamolodchikov proposed a three-dimensional generalization of YBE based on scattering of straight strings. A lattice interpretation was later given by Bazhanov and Stroganov [7] and by Baxter [8]. The work [6–8] produced a spectral parameter dependent equation (with three 'spectral angles' related by rules of spherical

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trigonometry). As for YBE one has also a spectral parameter independent formulation, which can be taken as a special limit or may be considered in its own right. We will discuss here only this constant form of Zamolodchikov's tetrahedron equation (ZTE), which is

$$\mathcal{R}^{k_1k_2k_3}_{j_1j_2j_3}\mathcal{R}^{l_1k_4k_5}_{k_1j_4j_5}\mathcal{R}^{l_2l_4k_6}_{k_2k_4j_6}\mathcal{R}^{l_3l_5l_6}_{k_3k_5k_6} = \mathcal{R}^{k_3k_5k_6}_{j_3j_5j_6}\mathcal{R}^{k_2k_4l_6}_{j_2j_4k_6}\mathcal{R}^{k_1l_4l_5}_{j_1k_4k_5}\mathcal{R}^{l_1l_2l_3}_{k_1k_2k_3} \tag{3a}$$

or in the shorthand notation

$$\mathcal{R}_{123}\mathcal{R}_{145}\mathcal{R}_{246}\mathcal{R}_{356} = \mathcal{R}_{356}\mathcal{R}_{246}\mathcal{R}_{145}\mathcal{R}_{123}. \tag{3b}$$

Now there are N^{12} equations for N^6 variables; even for the simplest non-trivial case of $N = 2$ there are 4096 equations for 64 variables.

There is a natural method of generalizing this to higher dimensions, see [7, 9]. The hierarchy so obtained is not just formal: in the following we show how each solution of a higher level equation yields solutions for the lower level, and how each lower level solution can be used to get solutions at higher level.

There is also another generalization to three dimensions by Frenkel and Moore [10]:

$$\mathcal{F}_{123}\mathcal{F}_{124}\mathcal{F}_{134}\mathcal{F}_{234} = \mathcal{F}_{234}\mathcal{F}_{134}\mathcal{F}_{124}\mathcal{F}_{123}. \tag{4}$$

Here there are only four different subspaces on which the matrices act. Carter and Saito have shown [11] how both of these fit into a generalized sequence of higher dimensional equations, ZTE and Frenkel–Moore equation are just located on different rays starting from YBE. In this letter we consider only ZTE.

Before discussing the solutions it is useful to recall the symmetries of the tetrahedron equations. This is important for classifying the solutions, for there is no point in repeating a solution in a form that can be obtained by one of the allowed transformations.

The symmetries of (3) are basically the same as those of YBE. First of all there is the invariance under continuous transformations

$$\mathcal{R} \to \kappa(Q \otimes Q \otimes Q)\mathcal{R}(Q \otimes Q \otimes Q)^{-1} \tag{5}$$

where Q is a non-singular $N \times N$ matrix and κ a non-zero number.

There are also discrete symmetries:

$$\mathcal{R}^{lmn}_{ijk} \to \mathcal{R}^{ijk}_{lmn} \tag{6a}$$

$$\mathcal{R}^{lmn}_{ijk} \to \mathcal{R}^{l+s,m+s,n+s}_{i+s,j+s,k+s} \qquad (\text{indices mod}\, N) \tag{6b}$$

$$\mathcal{R}^{lmn}_{ijk} \to \mathcal{R}^{nml}_{kji}. \tag{6c}$$

In writing out the triple-indexed object we use the usual matrix notation and the connection is

$$\mathcal{R}^{l+(m-1)N+(n-1)N^2}_{i+(j-1)N+(k-1)N^2} = \mathcal{R}^{lmn}_{ijk}.$$

In this notation (6*a*) corresponds to the usual matrix transposition. For $N = 2$, $s = 1$ (6*b*) followed by (6*a*) corresponds to transposition across the auxiliary diagonal;

transformation (6*c*) corresponds to the simultaneous exchange of the following columns (and rows): $2 \leftrightarrow 5$, and $4 \leftrightarrow 7$.

In the following we need a notation for traces of multi-index matrices, we use a square bracket to denote the location of the traced index, e.g. $(\mathcal{R}[2])_{ij}^{kl} = \sum_m \mathcal{R}_{imj}^{kml}$.

Let us now assume that we have a non-singular solution of (3). Multiplying (3*a*) by $(\mathcal{R}^{-1})_{l_1 l_2 l_3}^{j_1 j_2 j_3}$ and summing over the repeated indices yields (2) for $\mathcal{R}[1]$ with renumbering of indices. The same result is obtained for $\mathcal{R}[3]$ with $(\mathcal{R}^{-1})_{l_3 l_5 l_6}^{j_3 j_5 j_6}$. One can obtain an even stronger result. For that we need the following:

Definition. Three double-indexed matrices (A, M, B) form an *associated triple* of Yang–Baxter matrices (AT) if the following equations hold

$$\begin{aligned} A_{12}A_{13}A_{23} &= A_{23}A_{13}A_{12} \\ M_{12}M_{13}A_{23} &= A_{23}M_{13}M_{12} \\ B_{12}M_{13}M_{23} &= M_{23}M_{13}B_{12} \\ B_{12}B_{13}B_{23} &= B_{23}B_{13}B_{12}. \end{aligned} \tag{7}$$

Trivially (R, R, R) is an AT if R is a constant solution of (2), but more interestingly, if $R(x)$ is a solution of (1), then $(R(0), R(x), R(0))$ is also an AT, as can be readily seen by substituting $u = 0$ and/or $v = 0$ into (1). This does not exhaust the solutions, for example (P, M, P), where P is the permutation matrix, is an AT for an arbitrary M.

Let us now return to ZTE. Above we showed that if $\mathcal{R}$ is non-singular then $\mathcal{R}[1]$ and $\mathcal{R}[3]$ satisfy (2). If also $\mathcal{R}^{t_1}$ and $\mathcal{R}^{t_3}$ (transpose on the first and third index, respectively) are non-singular, then after multiplying (3*a*) by $((\mathcal{R}^{t_1})^{-1})_{j_1 l_4 l_5}^{l_1 j_4 j_5}$ and by $((\mathcal{R}^{t_3})^{-1})_{l_2 l_4 j_6}^{j_2 j_4 l_6}$ one finds that $(\mathcal{R}[1], \mathcal{R}[2], \mathcal{R}[3])$ must be an AT. This fact can be used in limiting the ansatz for ZTE.

If the same method is applied to (2) one gets the condition of commutativity for its trace matrices. For constant solutions the only non-trivial condition is that R_{im}^{jm} and R_{mi}^{mj} commute. (It should be noted here that non-singularity is a necessary requirement and there is a singular solution ($R_{H1.5}$ of [13]) for which these trace matrices do not commute.)

An interesting open problem is to see if the above works with spectral parameters. At each level the spectral parameters live in different spaces, but there is probably a natural projection which is necessary for obtaining the correct equation for the trace matrices.

Above we showed how each non-singular solution of (3) in dimension N yields an AT of the same dimension. We will now show how each such AT yields a solution of (3) of dimension N^2. This is a generalization of [12, 11] where the AT (R, R, R) was used.

Let

$$\mathcal{R}_{j_1 j_2 j_3}^{l_1 l_2 l_3} \equiv \mathcal{R}_{(j_1^a j_1^b)(j_2^a j_2^b)(j_3^a j_3^b)}^{(l_1^a l_1^b)(l_2^a l_2^b)(l_3^a l_3^b)} := A_{j_1^a j_2^a}^{l_1^a l_2^a} M_{j_1^b j_3^a}^{l_1^b l_3^a} B_{j_2^b j_3^b}^{l_2^b l_3^b} \tag{8}$$

then we have

$$
\begin{aligned}
&\mathcal{R}^{k_1k_2k_3}_{j_1j_2j_3}\mathcal{R}^{l_1k_4k_5}_{k_1j_4j_5}\mathcal{R}^{l_2l_4k_6}_{k_2k_4j_6}\mathcal{R}^{l_3l_5l_6}_{k_3k_5k_6}\\
&\quad= (A^{k_1^a k_2^a}_{j_1^a j_2^a}\, M^{k_1^b k_3^a}_{j_1^b j_3^a}\, B^{k_2^b k_3^b}_{j_2^b j_3^b})(A^{l_1^a k_4^a}_{k_1^a j_4^a}\, M^{l_1^b k_5^a}_{k_1^b j_5^a}\, B^{k_4^b k_5^b}_{j_4^b j_5^b})\\
&\qquad\times (A^{l_2^a l_4^a}_{k_2^a k_4^a}\, M^{l_2^b k_6^a}_{k_2^b j_6^a}\, B^{l_4^b k_6^b}_{k_4^b j_6^b})(A^{l_3^a l_5^a}_{k_3^a k_5^a}\, M^{l_3^b l_6^a}_{k_3^b k_6^a}\, B^{l_5^b l_6^b}_{k_5^b k_6^b})\\
&\quad= [A^{k_1^a k_2^a}_{j_1^a j_2^a}\, A^{l_1^a k_4^a}_{k_1^a j_4^a}\, A^{l_2^a l_4^a}_{k_2^a k_4^a}][M^{k_1^b k_3^a}_{j_1^b j_3^a}\, M^{l_1^b k_5^a}_{k_1^b j_5^a}\, A^{l_3^a l_5^a}_{k_3^a k_5^a}]\\
&\qquad\times [B^{k_2^b k_3^b}_{j_2^b j_3^b}\, M^{l_2^b k_6^a}_{k_2^b j_6^a}\, M^{l_3^b l_6^a}_{k_3^b k_6^a}][B^{k_4^b k_5^b}_{j_4^b j_5^b}\, B^{l_4^b k_6^b}_{k_4^b j_6^b}\, B^{l_5^b l_6^b}_{k_5^b k_6^b}]\\
&\quad= [A^{k_2^a k_4^a}_{j_2^a j_4^a}\, A^{k_1^a l_4^a}_{j_1^a k_4^a}\, A^{l_1^a l_2^a}_{k_1^a k_2^a}][A^{k_3^a k_5^a}_{j_3^a j_5^a}\, M^{k_1^b l_3^a}_{j_1^b k_3^a}\, M^{l_1^b l_5^a}_{k_1^b k_5^a}]\\
&\qquad\times [M^{k_3^b k_6^a}_{j_3^b j_6^a}\, M^{k_2^b l_6^a}_{j_2^b k_6^a}\, B^{l_2^b l_3^b}_{k_2^b k_3^b}][B^{k_5^b k_6^b}_{j_5^b j_6^b}\, B^{k_4^b l_6^b}_{j_4^b k_6^b}\, B^{l_4^b l_5^b}_{k_4^b k_5^b}]\\
&\quad= (A^{k_3^a k_5^a}_{j_3^a j_5^a}\, M^{k_3^b k_6^a}_{j_3^b j_6^a}\, B^{k_5^b k_6^b}_{j_5^b j_6^b})(A^{k_2^a k_4^a}_{j_2^a j_4^a}\, M^{k_2^b l_6^a}_{j_2^b k_6^a}\, B^{k_4^b l_6^b}_{j_4^b k_6^b})\\
&\qquad\times (A^{k_1^a l_4^a}_{j_1^a k_4^a}\, M^{l_1^b l_5^a}_{k_1^b k_5^a}\, B^{l_4^b l_5^b}_{k_4^b k_5^b})(A^{l_1^a l_2^a}_{k_1^a k_2^a}\, M^{k_1^b l_3^a}_{j_1^b k_3^a}\, B^{l_2^b l_3^b}_{k_2^b k_3^b})\\
&\quad= \mathcal{R}^{k_3k_5k_6}_{j_3j_5j_6}\mathcal{R}^{k_2k_4j_6}_{j_2j_4k_6}\mathcal{R}^{k_1l_4l_5}_{j_1k_4k_5}\mathcal{R}^{l_1l_2l_3}_{k_1k_2k_3}
\end{aligned}
\tag{9}
$$

i.e. $\mathcal{R}$ satisfies (3). The third equality sign follows from the assumption that (A, M, B) forms an AT. Note that if we use the AT $(R(0), R(x), R(0))$ the spectral parameter of YBE plays now the role of an ordinary parameter in a constant solution of (3).

In the above construction the dimension of the space gets squared. There is also a method of getting solutions of the same dimension.

Let us assume that $\mathcal{R}_{123} = R_{12} \otimes m_3$. Then it is easy to see that $\mathcal{R}$ solves (3) if any of the following holds

(a) $m^2 = 0$: for example

$$
R = A \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
$$

is a solution for arbitrary A.

(b) $(m \otimes m)R = R(m \otimes m) = 0$: here we may nevertheless assume that $m^2 \neq 0$, thus for a non-trivial solution there is precisely one non-zero entry on the diagonal. A typical solution is

$$
R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & j \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
$$

(c) R is a solution of (2) and $[R, m \otimes m] = 0$: the last commutation condition is trivially true if $m = 1_{2\times 2}$, but depending on R it can have much more general solutions and provide non-trivial interaction.

The above works equally well for $\mathcal{R}_{123} = m_1 \otimes R_{23}$.

We have also searched directly for particular two-dimensional solutions that do not fit into the general form of given above. Since the full set has 4096 quadratic equations in 64 variables it is not yet feasible to find the complete solution as in [13]. We will now describe the results of some ansatze that worked well for the Yang–Baxter equation. The computations were done using the symbolic algebra language REDUCE 3.4 [14], the equations were analysed using the GROEBNER package [15] written for REDUCE.

Upper triangular ansatz. If the entries on the diagonal are = 1, then the only new solution is

$$\mathcal{R}_1 = \begin{pmatrix} 1 & q & p & k & d & c & b & a \\ 0 & 1 & 0 & p & 0 & d & 0 & b \\ 0 & 0 & 1 & q & 0 & 0 & d & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 1 & q & p & k \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & q \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

This is clearly a tetrahedron generalization of the Yang–Baxter solution $R_{H2.3}$ of [13].

We have also looked at solutions with arbitrary non-zero entires on the diagonal. This search is still open but so far we have not found anything interesting.

Bidiagonal ansatz. In [6] Zamolodchikov proposed an ansatz for obtaining a spectral parameter dependent solution. As shown in [9] it amounts to allowing non-zero entries only on the diagonal and on the auxiliary diagonal with certain symmetry relations. We take this bidiagonal without any additional relations, i.e.

$$\mathcal{R}_B = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 \\ 0 & b_2 & 0 & 0 & 0 & 0 & b_7 & 0 \\ 0 & 0 & c_3 & 0 & 0 & c_6 & 0 & 0 \\ 0 & 0 & 0 & d_4 & d_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_4 & e_5 & 0 & 0 & 0 \\ 0 & 0 & f_3 & 0 & 0 & f_6 & 0 & 0 \\ 0 & g_2 & 0 & 0 & 0 & 0 & g_7 & 0 \\ h_1 & 0 & 0 & 0 & 0 & 0 & 0 & h_8 \end{pmatrix}. \tag{11a}$$

To save space we write out only the two diagonals:

$$\mathcal{R}_B = [a_1, b_2, c_3, d_4, e_5, f_6, g_7, h_8; a_8, b_7, c_6, d_5, e_4, f_3, g_2, h_1]. \tag{11b}$$

First of all we have a purely diagonal solution

$$\mathcal{R}_2 = [a_1, b_2, c_3, d_4, e_5, f_6, g_7, h_8; 0, 0, 0, 0, 0, 0, 0, 0] \tag{12}$$

as we have for YBE ($R_{H3.1}$ of [13]).

If there are non-zero entries only on the auxiliary diagonal, there must already be relations among them, as follows

$$\mathcal{R}_3 = [0, 0, 0, 0, 0, 0, 0, 0; a_8, b_7, c_6, d_5, d_5, c_6, b_7, a_8] \tag{13}$$

cf $R_{H1.4}$ of [13].

We have also searched for all solutions where all entries on both diagonals are actually non-zero and for which the determinant of the matrix is also non-zero. This resulted in the following solutions:

$$\mathcal{R}_4 = [1,1,\epsilon_1,\epsilon_2,1,\epsilon_1\epsilon_2,\epsilon_2,\epsilon_2; q,q,\epsilon_2 q,\epsilon_1 q,q,\epsilon_1\epsilon_2 q,\epsilon_1 q,\epsilon_1 q] \tag{14}$$

and

$$\mathcal{R}_5 = [1,\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_1,\epsilon_1\epsilon_2\epsilon_3,\epsilon_3,\epsilon_1\epsilon_3; 1,-\epsilon_2\epsilon_3,-\epsilon_2,\epsilon_1\epsilon_2,-\epsilon_2\epsilon_3,\epsilon_1\epsilon_2\epsilon_3,\epsilon_1\epsilon_2,-\epsilon_1\epsilon_3]. \tag{15}$$

Here q is a free parameter and $\epsilon_i = \pm 1$, independently. Note however, that if all $\epsilon_i = +1$ then the solutions can in fact be diagonalized.

Within this ansatz there are numerous other solutions with some entries zero, let me just mention one:

$$\mathcal{R}_6 = [1,\xi^2,\xi,\epsilon_1,\xi^2,\xi\epsilon_1,\epsilon_2,\xi^2\epsilon_2; 1,0,0,0,0,0,0,0] \tag{16}$$

where $\xi^6 = 1$.

Other solutions. We have also searched for solutions for which $\mathcal{R}_{ijk}^{lmn} \neq 0$ only if $i+j+k-l-m-n=0$, but this ansatz did not yield any interesting non-singular solutions. One nice singular solution is

$$\mathcal{R}_7 = \delta_{j_1}^{l_1}\delta_{j_2}^{l_2}\delta_{j_3}^{l_3} - \delta_{j_1}^{l_3}\delta_{j_2}^{l_2}\delta_{j_3}^{l_1} + \delta_{j_1}^{l_3}\delta_{j_2}^{l_1}\delta_{j_3}^{l_2}. \tag{17}$$

There is a huge number singular solutions; here are just two that are 'arrow-like':

$$\mathcal{R}_8 = \begin{pmatrix} 0 & 0 & 0 & a_4 & 0 & a_6 & a_4 & a_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

and if the lower left hand corner is zero, then the other entries are free:

$$\mathcal{R}_9 = \begin{pmatrix} 0 & 0 & 0 & a_4 & 0 & a_6 & a_4 & a_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

In this letter we have shown that, contrary to the popular pessimistic view, ZTE does indeed have many solutions. Some of them are inherited from the YBE but even then they have some extra structure. As shown above, there are also genuinely new solutions with no such connection. (The algebraic aspects of these solutions will be discussed elsewhere [16].) Here we have only scratched the surface and many interesting solutions are still to be found. For example so far we have no genuinely new solutions where the parameters are related in a non-linear way. What we now need is a fruitful ansatz; probably it will come from physical applications.

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